# Midterm 2 Solutions

Each question is worth 5 points.

**Problem 1** *For each $n \in \mathbb{N}$, let $a_n$ be the number of permutations in $S_n$ whose cube is the identity permutation. Prove combinatorially that $a_{n+3} = a_{n+2} + (n+2)(n+1)a_n$ for every $n \in \mathbb{N}$.*

*Solution.* For each $n \in \mathbb{N}$, let $T_n$ be the set of permutations in $S_n$ whose cube is the identity. Let $(c_1\, c_2\, \dots\, c_\ell)$ be a cycle of length greater than 1 in the disjoint cycle decomposition of $\pi \in T_n$. Note that $\ell \neq 2$ as otherwise $\pi^3(c_1) = c_2$, and this cannot happen because $\pi^3$ is the identity permutation. In addition,

$$c_3 = \pi(c_3) = \pi^2(c_2) = \pi^3(c_1) = c_1,$$

which implies that $\ell = 3$. Thus, every cycle in the disjoint cycle decomposition of $\pi$ has length either 1 or 3. The subset of $T_{n+3}$ whose elements have $(n+3)$ as a cycle is in bijection with $T_{n+2}$, where the bijection consists in dropping the cycle $(n+3)$. On the other hand, we can construct any permutation in $T_{n+3}$ in which the cycle containing $n+3$ has length greater than one (that is, has length 3) as follows: construct the three-cycle $(i\ j\ n+3)$ containing $n+3$ in $(n+2)(n+1)$ different ways (choosing an ordered pair $(i, j)$ with $i \neq j$ from the set $[n+2]$), and then make the set $[n+2] \setminus \{i, j\}$ into a cycle type decomposition in $a_n$ different ways. Hence $a_{n+3} = a_{n+2} + (n+2)(n+1)a_n$, as desired. □

**Problem 2** *Suppose that we have $n$ distinguishable standard dice. A roll of the $n$ dice is called* complete *if each of the numbers $1, 2, 3, 4, 5, 6$ appears on the face of at least one die. For instance, if $n = 7$, then the roll $(5, 3, 1, 2, 4, 5, 6)$ is complete while the roll $(3, 5, 4, 3, 5, 1, 1)$ is not complete because 2 and 6 do not appear on any face. Let $d_n$ be the number of complete rolls of the $n$ dice. It is clear that if $n \in [5]$, then $d_n = 0$. Find a formula for $d_n$ for $n \geq 6$ (summation signs are allowed).*

*Solution.* For each $j \in [6]$, let $A_j$ be the set of rolls of the $n$ dice in which $j$ does not show (in any of the $n$ faces up). Observe that if $S$ is a $k$-subset of $[6]$, then $|\bigcap_{j \in S} A_j| = (6-k)^n$. Then it follows from the Sieve Method that

$$\Big|\bigcup_{n=1}^{6} A_n\Big| = \sum_{\emptyset \neq S \subseteq [6]} (-1)^{|S|+1} \Big|\bigcap_{j \in S} A_j\Big| = \sum_{k=1}^{6} \sum_{S \subseteq [6] : |S| = k} (-1)^{|S|+1} \Big|\bigcap_{j \in S} A_j\Big|$$

$$= \sum_{k=1}^{6} \binom{6}{k} (-1)^{k+1} (6-k)^n.$$

It is clear that a roll is complete precisely when it does not belong to any of the sets $A_j$. Since there are a total of $6^n$ possible outcomes for the rolls of the $n$ dice, the number of complete rolls is

$$6^n - \Big|\bigcup_{n=1}^{6} A_n\Big| = 6^n - \sum_{k=1}^{6} \binom{6}{k}(-1)^{k+1}(6-k)^n = \sum_{k=0}^{6} \binom{6}{k}(-1)^k(6-k)^n.$$

□

**Problem 3** *Let $(a_n)_{n\geq 0}$ be the sequence satisfying that $a_{n+1} = (n+1)a_n + 2\cdot(n+1)!$ for every $n \geq 0$ and $a_0 = 0$.*

1. *(3 pts) Find a closed formula (no summation signs allowed) for the exponential generating function of $(a_n)_{n\geq 0}$.*

2. *(2 pts) Find an explicit formula for $a_n$.*

*Solution.* 1. Let $F(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ be the exponential generating function of $(a_n)_{n\geq 0}$. From $a_0 = 0$ and $a_{n+1} = (n+1)a_n + 2\cdot(n+1)!$, we deduce that

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} a_{n+1}\frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty}(n+1)a_n\frac{x^{n+1}}{(n+1)!} + 2\sum_{n=0}^{\infty}(n+1)!\frac{x^{n+1}}{(n+1)!} \\ &= x\sum_{n=0}^{\infty} a_n\frac{x^n}{n!} + 2x\sum_{n=0}^{\infty} x^n = xF(x) + \frac{2x}{1-x}. \end{aligned}$$

Therefore $F(x) = \frac{2x}{(1-x)^2}$.

2. In order to find an explicit formula for $a_n$, we use the generalized binomial theorem:

$$\begin{aligned} F(x) = 2x(1-x)^{-2} &= 2x\sum_{n=0}^{\infty}\binom{-2}{n}(-x)^n \\ &= 2x\sum_{n=0}^{\infty}(-1)^n\frac{(-2)(-3)\cdots(-2-n+1)}{n!}x^n \\ &= 2x\sum_{n=0}^{\infty}\frac{(n+1)!}{n!}x^n = 2\sum_{n=0}^{\infty}(n+1)x^{n+1} \\ &= \sum_{n=1}^{\infty} 2nx^n. \end{aligned}$$

Thus, $a_n/n! = 2n$, which means that $a_n = 2n\cdot n!$. □

**Problem 4** *For each $n \in \mathbb{N}$, let $f_n$ be the number of ways to split a line of $n$ soldiers into nonempty sub-intervals, create a platoon out of each resulting sub-interval, and choose a commander in each platoon. Assume that $f_0 = 1$. Find a closed formula for the (ordinary) generating function of $(f_n)_{n\geq 0}$.*

*Solution.* Consider the sequence $(a_n)_{n\geq 0}$, where $a_0 = 0$ and $a_n$ is the number of ways to create a platoon with a leader out of a sub-interval consisting of $n$ soldiers. If $A(x)$ is the (ordinary) generating function of $(a_n)_{\geq 0}$, we obtain that

$$A(x) = \sum_{n=1}^{\infty} nx^n = x\sum_{n=0}^{\infty} nx^{n-1} = x\frac{d}{dx}\Big(\sum_{n=0}^{\infty} x^n\Big) = x\frac{d}{dx}\Big(\frac{1}{1-x}\Big) = \frac{x}{(1-x)^2}.$$

Let $B(x)$ be the (ordinary) generating function corresponding to building the trivial structure (that is, doing nothing) on the set of all created platoons. Then

$$B(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Hence if $F(x)$ is the ordinary generating function of $(f_n)_{n\geq 0}$, then it follows from the composition theorem for (ordinary) generating functions that

$$F(x) = \sum_{n=0}^{\infty} f_n x^n = B(A(x)) = \frac{1}{1-A(x)} = \frac{(1-x)^2}{(1-x)^2 - x}.$$

□